Section 19: Compact Operators and Orthonormal Families

A question arises about how compact operators map orthonormal families. If T is compact and normal and $\{\varphi_p\}_{p=1}^{\infty}$ is the sequence of orthonormal eigenvalues, then $\lim_{n\to\infty}\{T\varphi_n\} = \lim_{n\to\infty}\{\lambda\varphi_n\} = 0$. What if T is compact, but not necessarily normal (or self adjoint) and $\{\varphi_p\}_{p=1}^{\infty}$ is an orthonormal family. Must $\lim_{n\to\infty}\{T\varphi_n\} = 0$?

Theorem 39 Suppose that T is compact and $\{\varphi_p\}_{p=1}^{\infty}$ is an orthonormal family. Then $\lim_{n\to\infty}\{T\varphi_n\} = 0$.

<u>Proof:</u> Suppose not. There is a subsequence $\{\varphi_{u(p)}\}_{p=1}^{\infty}$ such that $|T\varphi_{u(p)}| \geq \varepsilon$ for some $\varepsilon \geq 0$. Since $\{\varphi_{u(p)}\}_{p=1}^{\infty}$ is bounded and T is compact, there is a subsequence of $\{T\varphi_{u(p)}\}_{p=1}^{\infty}$ that converges and has limit, say, $v \neq 0$. Then,

$$\lim_{n\to\infty} < T\varphi_{u(n)}, v> = < v, v > \neq 0.$$

But, also,

$$\lim_{n\to\infty} < T\varphi_{u(n)}, v> = \lim_{n\to\infty} < \varphi_{u(n)}, T^*v > = 0$$

because these last are terms in the Fourier expansion of

$$T^*v = \Sigma_p < T^*v, \varphi_p> \varphi_p.$$

The terms of this sum must go to zero. This gets a contradiction.

Remark The ideas around Theorem 38 provides an easy characterization of when operators commute.

Theorem 40 Supose that A and B are compact, normal operators. These are equivalent:
(a) AB = BA, and

(b) There is a maximal orthonormal family $\{\varphi_p\}_{p=1}^{\infty}$ which are eigenvectors for A and for B.

Proof. If (b) holds, this is clear from the representation of Theorem 36. Suppose that (a) holds and λ is an eigenvalue of A. Let S be the subspace of vectors x such that $Ax = \lambda x$. Because A and B commute, B maps S into S. Hence there is a sequence of eigenvectors for B that spans S. But, each of these is an eigenvector for A corresponding to λ. This process is symmetric in A and B. The representation of Theorem 36 completes the result.

Remarks

(1) if λ is a nonzero eigenvalue for BA, then it is an eigenvalue for AB. To see this, suppose $BAx = \lambda x$. Then

$$(AB)\, Ax = A\lambda x = \lambda\, Ax.$$

Thus, λ is a non zero eigenvalue for AB.

2) Some use this last result to characterize normal operators this way: an operator is normal if and only if it and its adjoint can be "simultaneously diagonalized."

(3) This course has investigated the representation of linear transformations as $\sum_{p=1}^{\infty} \lambda_p \langle x, \varphi_p\rangle \varphi_p$. This representation gives insight as to the nature of linear transformations. We have found the representation is appropriate for compact, self-adjoint and normal operators From examples, we have seen that it gives an understanding to bounded, even if not compact operators, and even to unbounded operators on a Hilbert space. The representation should give insight and unification to some of the ideas that are encountered in a study of integral equation, Green's functions, partial differential equations, and Fourier series.

Assignment

(1) Find the eigenvalues for

$$R(x) = \sum_{p=1}^{\infty} \langle x,\varphi_p\rangle \varphi_{p+1} \text{ and for } L(x) = \sum_{p=1}^{\infty} \langle x, \varphi_{p+1}\rangle \varphi_p.$$

(One of these has no eigenvalue and the other has every number in the unit disk as an eigenvalue.)

(2) Do the weighted left shift and right shift have a representation in the simple paradigm?

Maple Remark: The finite dimensional analogue to the right shift and the left shift might be explored as follows:

```
> with(linalg):
> R:=array([[0,0,0,0],[1,0,0,0],[0,1,0,0],[0,0,1,0]]);
  L:=array([[0,1,0,0],[0,0,1,0],[0,0,0,1],[0,0,0,0]]);
```

Looked at this way, we see that right-shift and left-shift are infinite dimensional analogues of nilpotent operators. We should check to confirm what are the eigenvalues and eigenvectors of these two.

```
> eigenvects(R); eigenvects(L);
```

There is a new idea that should be brought up here. One could define "generalized eigenvectors" for A as vectors v for which there is a number λ such that

$$(A - \lambda I)v \neq 0$$

but for which

$$(A - \lambda I)^2 v = 0.$$

This would be a generalized eigenvector v of rank 2. One could define a generalized eigenvector of rank k.

Here's an example:

```
> A:=array([[1,1,2],[0,1,3],[0,0,2]]);
> charpoly(A,x);
> eigenvects(A);
```

Note that 1 is an eigenvalue of multiplicity 2, but has only one eigenvector. We go looking for one generalized eigenvector of rank 2. To that end, we find the nullspace of $(A-I)^2$:

```
> AmIdnty:=evalm((A-diag(1,1,1))&*(A-diag(1,1,1)));
> nullspace(AmIdnty);
```

Thus, we have that

$$(A-2\ I)\{5,3,1\} = \{0,0,0\},$$

$$(A-1\ I)\{1,0,0\} = \{0,0,0\},$$

and

$$(A-1\ I)^2\{0,1,0\}=\{0,0,0\}.$$

Several questions come to mind:

(1) What are the generalized eigenvectors for R and L above?

(2) How do these fit into the context and structure for the paradigm presented in these notes?

Section 20: The Most General Paradigm: A Characterization of Compact Operators

The paradigm that has been sugested in these notes is applicable for compact and normal operators. This is a fairly satisfactory state of affairs. Yet, the simple matrices

$$\begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

do not fit into that situation. We will push the representation one more time. In addition to the satisfaction of having a decompositon that is applicable to those two matrices, we will be able to obtain the Fredholm Alternative Theorems for mappings with less hypothesis.

Lemma 41 (1) If A is bounded and B is compact then AB is compact.
(2) If A is compact and B is bounded then AB is compact.
(3) If T is compact, then T^* is compact.
(Hint: since T is compact, then TT^* is compact and $<TT^*x,x> = |T^*x|^2$.)

Theorem 42. Suppose that T is a compact operator from E to E. There are maximal orthonormal families $\{\Phi_p\}$ and $\{\Theta_p\}$ and a non-increasing number sequence $\{\lambda_p\}_{p=1}^{\infty}$ such that $\lim_p \lambda_p = 0$, and if x is in E, then

$$Tx = \sum_{p=1}^{\infty} \lambda_p < x, \Phi_p > \Theta_p.$$

Moreover, the convergence is in the norm of BLT.

<u>Proof:</u> Suppose that T is compact. First, T* is bounded since T is. To see this,

$$|T^*x|^2 = <T^*x,T^*x> = <TT^*x,x> \leq ||T|| \; |T^*x| \; |x|,$$

so that $|T^*x| \leq ||T|| \; |x|$. Now, knowing that T* is bounded and T is compact, we can get that T*T is compact and it is selfadjoint. Moreover, $<T^*Tx, x> \geq 0$ so that all the eigenvalues of T*T are nonnegative. Arrange all the eigenvalues in decreasing order. We have

$$T^*Tx = \sum_p \mu_p <x, x_p> x_p.$$

For each n such that $\mu_n \neq 0$, let

$$y_n = T(x_n)/\sqrt{\mu_n}.$$

Then $< y_n, y_m> = <Tx_n, Tx_m>/\sqrt{\mu_m \mu_n} = <T^*Tx_n,x_m>/\sqrt{\mu_m \mu_n}$

$$= \sqrt{\frac{\mu_n}{\mu_m}} <x_n,x_m> = 0.$$

Thus, $\{y_p\}$ is orthogonal, even orthonormal. Extend it to be a *maximal*.

Then $$T(x_p) = \sqrt{\mu_p}\, y_p$$

even if $\mu_p = 0$. Suppose that

$$x = \Sigma_p < x, x_p > x_p.$$

$Tx = T(\Sigma_p < x, x_p > x_p) = \Sigma_p <x,x_p> Tx_p = \Sigma_p \sqrt{\mu_p} <x,x_p> y_p,$

To see that this sum converges in the BLT norm,

$$| \Sigma_{n+1} \sqrt{\mu_p} <x,x_p> y_p |^2 = \Sigma_p \mu_p \; |<x,x_p>| < \mu_{p+1} \; |x|^2$$

Assignment

(20.1) Perhaps you will agree that applying this decomposition to the matrices

$$\begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

is irresistible. Note that this is different from the decomposition which we had in the first of these notes.

(20.2) With T as in Theorem 42, What is T*?

MAPLE Remark; We will get the generalized paradigm for a matrix T that is not normal.

```
> with(linalg):
> T:=array([[0,1,0],[0,0,0],[0,0,2]]);
```

We form the self-adjoint matrix A*A.

```
> A:= evalm(transpose(T) &* T);
> eigenvects(A);
> x[1]:=vector([0,1,0]); x[2]:= vector([0,0,1]); x[3]:= vector([1,0,0]);
> y[1]:=evalm(T &* x[1]/1); y[2]:=evalm(T &* x[2]/2);y[3]:=evalm(T &*
x[3]);
> y[3]:=[0,1,0];
```

The proposal is that $T(u) = 1 <u,x_1> y_1 + 2 <u, x_2> y_2 + 0 <u,x_3> y_3$. We check this.

```
> u:=vector([a,b,c]);
> evalm(T &* u);
> evalm(1*innerprod(u,x[1])*y[1]
                        + 2*innerprod(u,x[2])*y[2]
                                          + 0*innerprod(u,x[3])*y[3]);
```